



# “Recursion: Smaller Sibling Pyramids”

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## Lesson Focus

Recursion, Iteration (Looping), and Concurrency. In the first of two sessions (at most an hour each), students are asked to calculate a simple summation by themselves, based on a procedure they are given. Then, through a guided role-playing procedure, students are asked to do the same problem by pushing a sub-problem off onto a ‘little sibling’. In the second session, they use a divide-and-conquer approach to understand a simple formula for summation. During this session they also talk about the big ideas behind these three problem solving methods.

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## Age Levels

Recommended for ages 8 – 10  
Also appropriate for ages 10 – 13

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## Objectives

Introduce students to:

- ✦ how arithmetic sequences solve real world problems
  - ✦ tail-end recursive algorithms for arithmetic series
  - ✦ a divide and conquer approach that leads to a simple formula
  - ✦ informal ideas about time complexity.
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## Anticipated Learner Outcomes

Students will be able to describe how to solve an arithmetic sequence summation problem:

- ✦ by doing it again and again (non-concurrent iteration)
  - ✦ with a smaller sibling (tail-end recursively)
  - ✦ articulate that both methods take the same number of steps, but recursion is less work for the individual
  - ✦ divide and conquer has a surprising outcome – namely a formula that can be calculated in only a few steps.
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## Alignment to Curriculum Frameworks

See attached curriculum alignment sheet.

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## Internet Connections

These are for teacher reference:

- ✦ <http://www.regentsprep.org/regents/math/algtrig/atp2/sequencewordpractice.htm>
- ✦ [http://www.algebra.com/lessons/lesson.aspx?file=Algebra\\_SeqSeriesApps.xml](http://www.algebra.com/lessons/lesson.aspx?file=Algebra_SeqSeriesApps.xml)
- ✦ [http://www.cpm.org/pdfs/state\\_supplements/Sequences.pdf](http://www.cpm.org/pdfs/state_supplements/Sequences.pdf)

These are for students:

- ✦ [http://en.wikipedia.org/wiki/Human\\_pyramid](http://en.wikipedia.org/wiki/Human_pyramid)
- ✦ [http://en.wikipedia.org/wiki/Human\\_pyramid#/media/File:Escala\\_de\\_9\\_Falcons\\_Vil\\_afranca.jpg](http://en.wikipedia.org/wiki/Human_pyramid#/media/File:Escala_de_9_Falcons_Vil_afranca.jpg)

- ✦ [http://en.wikipedia.org/wiki/Human\\_pyramid#/media/File:Otago\\_Dancers\\_human\\_pyramid\\_2.jpg](http://en.wikipedia.org/wiki/Human_pyramid#/media/File:Otago_Dancers_human_pyramid_2.jpg)
  - ✦ <https://www.youtube.com/watch?v=mkD4CrWgFpl> (Cat in the Hat Comes Back)
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### Recommended Reading

- ✦ "The Cat in the Hat Comes Back", Dr. Seuss, Random House Children's Books, 1958. Although clearly below anticipated grade level, this is an outstanding book for young students to begin thinking about recursion and its ramifications.
  - ✦ Your school math textbook series will typically include examples of patterns and functions. Please choose grade-appropriate materials from your math text or from lower grade level texts.
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### Optional Writing Activity

- ✦ This activity introduced the idea of how to efficiently calculate an arithmetic series, such as  $1+2+3+4$ . This could be used to calculate the simple human pyramid where one person is added as support for each layer. Invent your own problem that produces a different arithmetic pattern such as  $1,5,9,13,17$ . Ask someone in your class to solve it by simple addition, by recursion, and to see if they can come up with a formula based on divide and conquer.

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## For Teachers: Teacher Resources

### ◆ Lesson Objectives

- ✦ Notice that some real-world problems can be solved by recognizing a pattern that produces an arithmetic series and perhaps a simple formula.
- ✦ Simple addition and recursion via a 'smaller sibling' both describe an algorithm for summing the values in a series. These algorithms take the same number of calculations, however recursion requires less work for the individual.
- ✦ By thinking about the problem differently, namely using 'divide and conquer', a simple calculation significantly reduces the amount of work required.

### ◆ Materials

- ✦ Access to videos on line. Download and cache ahead of time if necessary.
- ✦ Have sufficient copies of the worksheets below for all students to participate in a recursive calculation at least twice.
- ✦ Tokens (coins flat beads, etc.) for the divide and conquer activity, ideally 50 per teams of 2 – 4, but at least 21 per team.
- ✦ Sharpened pencils.

### ◆ Procedure

#### Session 1:

1. Talk about the human tower phenomenon. Watch the video <http://www.thehumantower.com/>. A simple human tower consists of layers where each layer has one less person than the one below. Pose the puzzle: how many people need to be recruited to create a simple tower 10 rows high?
2. Introduce the concept of an arithmetic series using Worksheet 1. Make sure they understand that this is the simplest and most stable tower. Individually or in pairs, calculate the answer for a 10-row-high tower, just by counting the number of people in the diagram. Ask them how many calculations they had to make. Explain that a calculation costs money, and the coach has to reduce the amount spent on calculating. Is there a better way to count? Introduce the idea that one person needs two people to support her, and those two people need three people, etc. So a ten-row-high tower is  $1+2+3+4$  etc. How many calculations does this take? Talk to them about the difference in cost and effort to make a pyramid 100 acrobats high. Don't have them calculate it, only have them think about what this would mean.
3. Remind students that recursion is a problem solving technique used on hard problems to make them easier. An example from children's literature (for kids much younger than these are) is "The Cat in the Hat Came Back". It is worth reading the story to them, have them read it to each other, or listen to the YouTube link given in the Internet resources. Recursion requires you to focus on a manageable part of a problem, such as doing a single calculation (e.g. an addition), and then giving a slightly less hard problem to some one else.

Worksheet 2 provides 6 students with the opportunity to solve the acrobat problem through a kinesthetic recursion exercise. Divide your class in a way that everyone

gets to participate, either by partnering students or running more than one recursion simulation as described below.

Work out the simulation on your own, so that you can direct the activity. Your students may need to work through it several times to get the hang of it, and once they do, perhaps have them calculate a larger pyramid. After the students have solved the problem, have them count how many additions they did. Help them notice that they did one addition for each 'smaller sibling'. They can count the number of siblings it took rather than the number of calculations.

### **Recursion Simulation for a six row pyramid**

- a. Assign one student to be the single person at the top of the pyramid. She needs to find out how many people are needed below her to make 5 rows below her. Help her fill out worksheet 2.
- b. Direct her to select a 'smaller sibling' to calculate that number for her. She should tell her smaller sibling that she will need 2 people below her. When the sibling responds with a result, the student at the top adds 1 (her row) to that number and declares the total number of people required.
- c. Help the first smaller sibling fill out his form. His big sibling told him the number of people required in his row and how many rows to go. He should ask a little sibling for help by giving that sibling the number of people in his row, and how many rows to go (one less than his number of rows). He can 'rest' until his little sibling produces a result. Talk to the students about how this is exactly what happens in a computer when one process waits for another. They 'put their computer to sleep' when all its applications are still 'alive' but waiting for something to happen.
- d. Help each subsequent sibling figure out whether he or she needs help from a smaller sibling. When a student is told she has no rows below her, she simply responds only to her big sibling with the number of people in her row. Each sibling 'wakes up' and adds her row count to the one from her little sibling until everyone up to the single person at the top adds their row count.

### **Session 2**

1. Remind the students of the acrobat problem. Have them play the recursion game again.
1. Another computing approach to efficiently doing the calculations is to 'divide and conquer'. This is illustrated with the flipped over human pyramid on Worksheet 3. The series can be calculated by a formula:  $(n+1)(n/2)$ . Looking at the flipped pyramid you can see how we can divide the pyramid into two halves. By flipping the top half and adding those members to the rows in the bottom half we can quickly calculate that we now have 3  $(n/2)$  rows, each of which contains 7  $(n+1)$  people.
2. Arrange teams of 2 – 4 and give each team enough tokens to build a tower of at least 6, and preferably 9 rows. Have them lay out a pyramid of 6 rows on a flat surface. (Do not have them build an actual pyramid.) Walk them through the idea of flipping the top half, and have them do it with their tokens.
3. Challenge the groups to come up with a formula for describing the number of people inside the parallelogram of the flipped pyramid. Have them create another

pyramid and see if the formula still works. The lesson here is not to memorize  $(n-1)n/2$  but rather to understand that you can sometimes find a very simple solution through divide-and-conquer.

4. Go back to the original problem or another simple arithmetic series. Lead a discussion about how it is really about how you think about a problem. Sometimes recursion gives you a good way to express a problem concisely, and sometimes a simple formula can be found that is more efficient. Computer scientists look for recursive solutions because these often lead to insights in how to solve a problem very efficiently. Explain that this is the tip of the iceberg (or pyramid), and that other kinds of recursive solutions capture wonderful natural phenomena.

◆ **Time Needed**

- ✦ 2 sessions, at most 1 hour each. It is important to leave sufficient time in the second session for the fourth exercise. If the activity is done in a single session, then skip Worksheet 3.

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## Student Resource: Problem Solving with Recursion

Human beings are very good at finding creative solutions to problems. But humans are easily distracted and make mistakes when they have to do a boring task. We created arithmetic rules so we could sometimes take short cuts to help us solve problems more quickly. For example, if you want to know how many students are in your school, you could line them all up and count them. But somebody would probably ask to go to the restroom or the nurse, and whoever was counting would lose track. A shortcut would be to ask the teachers for the number of students in their classes, and then add up those numbers.

Computer scientists write *algorithms* using very creative short-cuts so that the computer can figure out the answer more quickly. An algorithm is a list of instructions that someone else can follow. Writing an algorithm often requires thinking about a problem from a particular point of view. One very powerful point of view is called *recursion*. A nice way to think about recursion was invented by a computer scientist named Seymour Papert. The idea is that if you have a problem that you know will take a long time to figure out, you might want to ask some one to help you by having them solve a smaller problem for you. Mathematicians call this asking a 'smaller sibling'. This is not quite the Tom Sawyer approach to painting a fence, but it is close. No matter what, the idea is to find a good, easier way to list the steps that lead to a solution.

In this lesson you will learn about how problems that can be solved using something mathematicians call *arithmetic series*. You may have seen these in math class, where you had to predict what comes next:

1,2,3,4,?  
5,8,11,14,?  
19,?,25,28

There are many kinds of problems where you want to do a calculation with a series. For example:  $5+8+11+14 = ?$ . You could, of course, just do the addition with or without a calculator, but the odds are that unless you really like to add sums, you will be distracted and make a mistake along the way, especially if the series is long.

Calculating the number of people required to create a human pyramid is a problem that requires an arithmetic series. This is a good time to watch the first video. In the worksheets you will be asked to calculate the number of people required for the simplest kind of pyramid as shown in these pictures. The idea is quite simple: to make a pyramid with two rows you need one person on top of two other people. Three rows require a person on top of two people, who are on top of three people. Each additional row needs one more person below it. (You could of course do this with blocks, but it's more interesting using people.)

You will also learn about another technique called 'divide-and-conquer' that can sometimes help you find very efficient solutions to problems.

# Recursion: Smaller Sibling Pyramids

## Student Worksheet 1: How Many People in the Pyramid?

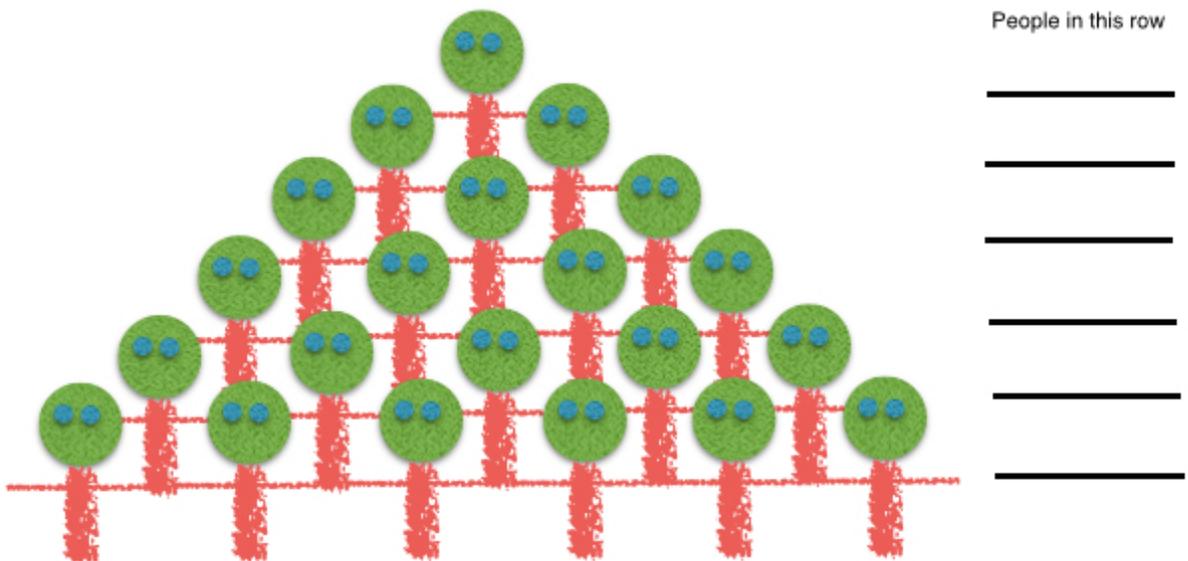
**The Problem:** A human pyramid coach wants to create a simple tower, 6 rows high. Each person needs to stand on a shoulder of one person below. See the illustration below. Put differently, each row consists of one more person than the row above.

1) Count how many people are in the pyramid: \_\_\_\_\_

2) Now take a short cut. How many people in the top row? And the next row down, and the next row down, etc.? Can you write the number in each row without counting?

What is the rule for knowing how many people are in the row below the row you are in?  
Hint: one number per row: \_\_\_\_\_

Which takes fewer calculations: to just count each head, or to add up the number in the rows?



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## Student Worksheet 2:

### Get help from a little sibling?

Follow your teacher's instructions to complete this sheet. This will be your notepad to use when everyone adds their totals together. Check which one applies to you when you are called on. Be sure to properly fill out the form, or the calculation will not be correct.

\_\_\_\_\_ **I am the person at the top.**

Number of people in my row: 1

My teacher said I should calculate a pyramid with \_\_\_\_\_ rows.

I will get help from a little sibling named \_\_\_\_\_.

I've told my little sibling that her row will need one more person than I have in my row, and she will need to calculate one less number of rows.

I can go to sleep until she wakes me up.

I will add the number of people in my row to the result given to me by my little sibling. The result is: \_\_\_\_\_.

I will give this result to my teacher.

\_\_\_\_\_ **I am a little sibling to \_\_\_\_\_.**

My big sibling told me that the number of people in my row is \_\_\_\_\_.

My big sibling said we should calculate a pyramid with \_\_\_\_\_ rows.

I will get help from a little sibling named \_\_\_\_\_.

I've told my little sibling that her row will need one more person than I have in my row, and she will need to calculate one less number of rows.

I can go to sleep until she wakes me up.

I will add the number of people in my row to the result given to me by my little sibling. The result is: \_\_\_\_\_.

I will wake up my big sibling and tell her my result.

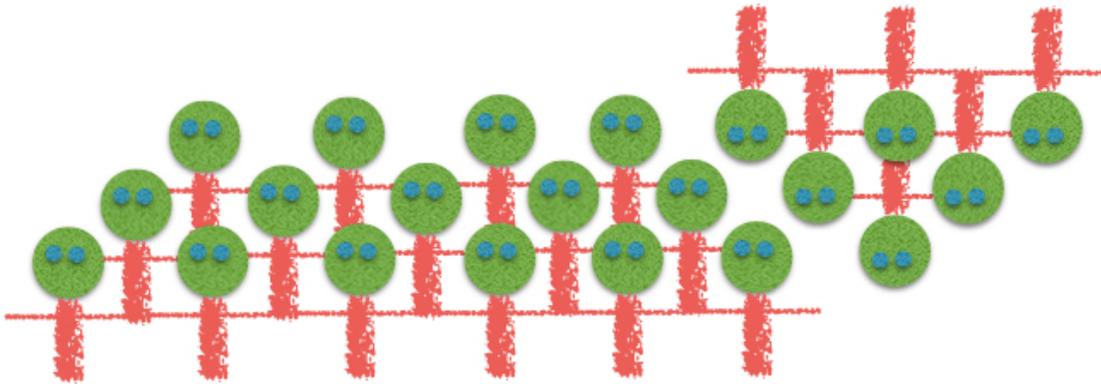
# Recursion: Smaller Sibling Pyramids

## Student Worksheet 3:

### Divide and Conquer to Find a Formula

Your teacher will give your team a collection of tokens. Pretend that they are acrobats. Lay out a pyramid of 6 rows on the table. Don't stack them on top of each other, just lay them out so you can easily see all of the rows.

Your teacher will also demonstrate how to flip the top half of the pyramid. Here is an illustration of what it should look like. (Of course you have tokens, not acrobats.)



How many rows do you have now? \_\_\_\_\_

How many people in each row? (The upside down guys count!) \_\_\_\_\_

Create a formula for calculating the number of people in the whole pyramid:

Using your remaining coins, build pyramids of other sizes. Flip the top, count the rows and the people in the rows. Build some that are smaller and some larger. Does your formula work for them as well? What happens if you have an odd number of rows (say 5)?

Number of Rows	Number of People in the Row	Total Number of People

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## For Teachers: Alignment to Curriculum Frameworks

Note: All lesson plans in this series are aligned to the Computer Science Teachers Association K-12 Computer Science Standards, and if applicable also the U.S. Common Core State Standards for Mathematics, the U.S. National Council of Teachers of Mathematics' Principles and Standards for School Mathematics, the International Technology Education Association's Standards for Technological Literacy, the U.S. National Science Education Standards and the U.S. Next Generation Science Standards.

### ◆National Science Education Standards Grades K-4 (ages 4-9)

#### **CONTENT STANDARD E: Science and Technology**

As a result of activities, all students should develop

- ✦ Understanding about science and technology

### ◆National Science Education Standards Grades 5-8 (ages 10-14)

#### **CONTENT STANDARD E: Science and Technology**

As a result of activities, all students should develop

- ✦ Understandings about science and technology

### ◆Next Generation Science Standards & Practices Grades 3-5 (ages 8-11)

#### **Practice 2: Generating and Using Models**

- ✦ Use a model to test cause and effect relationships or interactions concerning the functioning of a natural or designed system.

#### **Practice 6: Constructing Explanations and Designing Solutions**

- ✦ Use evidence (e.g., measurements, observations, patterns) to construct or support an explanation or design a solution to a problem

### ◆Next Generation Science Standards & Practices Grades 6-8 (ages 11-14)

#### **Practice 2: Generating and Using Models**

- ✦ Develop and/or use a model to generate data to test ideas about phenomena in natural or designed systems, including those representing inputs and outputs, and those at unobservable scales.

#### **Practice 5: Using Mathematics and Computational Thinking**

- ✦ Use mathematical representations to describe and/or support scientific conclusions and design solutions.

#### **Practice 6: Constructing Explanations and Designing Solutions**

- ✦ Construct an explanation using models or representations

### ◆Principles and Standards for School Mathematics Grades 3-5 (ages 8 - 11)

#### **Algebra**

- Represent and analyze mathematical situations and structures using algebraic symbols
  - ✦ Express mathematical relationships using equations.
- Use mathematical models to represent and understand quantitative relationships.
  - ✦ Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.

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## For Teachers:

### Alignment to Curriculum Frameworks

#### ◆ Principles and Standards for School Mathematics Grades 6-8 (ages 11 - 13)

##### **Algebra Standards**

- Represent and analyze mathematical situations and structures using algebraic symbols
  - ✦ Use symbolic algebra to represent situations and solve problems
- Use mathematical models to represent and understand quantitative relationships.
  - ✦ Model and solve contextualized problems using various representations such as graphs, tables and equations.

#### ◆ Principles and Standards for School Mathematics (all ages)

##### **Problem Solving Standards**

- ✦ Solve problems that arise in mathematics and other contexts

##### **Communication Standards**

- ✦ Communicate their mathematical thinking coherently and clearly to peers, teachers and others

##### **Representation**

- ✦ Use representations to model and interpret physical, social and mathematical phenomena

#### ◆ Common Core State Standards for School Mathematics Gr. 6-8 (ages 11-13)

##### **Operations & Algebraic Thinking Standard**

- Write and interpret numerical expressions.
  - ✦ CCSS.MATH.CONTENT.5.OA.A.1 Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

##### **Expressions & Equations Standard**

- Apply and extend previous understandings of arithmetic to algebraic expressions.
  - ✦ CCSS.MATH.CONTENT.6.EE.A.2.A - Write expressions that record operations with numbers and with letters standing for numbers.

#### ◆ Common Core State Practices & Standards for School Mathematics (all ages)

- ✦ CCSS.MATH.PRACTICE.MP1 Make sense of problems and persevere in solving them.
- ✦ CCSS.MATH.PRACTICE.MP4 Model with mathematics.

#### ◆ Standards for Technological Literacy - all ages

##### **Nature of Technology**

- ✦ Standard 2: Students will develop an understanding of the core concepts of technology

##### **The Designed World**

- ✦ Standard 17: Students will develop an understanding of and be able to select and use information and communication technologies

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## For Teachers: Alignment to Curriculum Frameworks

### ◆CSTA K-12 Computer Science Standards Grades 3-6 (ages 8-11)

#### 5.1 Level 1: Computer Science and Me (L1)

- ✦ Computational Thinking (CT)
  2. Develop a simple understanding of an algorithm (e.g., search, sequence of events, or sorting) using computer-free exercises.
  4. Describe how a simulation can be used to solve a problem.
- ✦ Collaboration (CL)
  3. Identify ways that teamwork and collaboration can support problem solving and innovation.

### ◆CSTA K-12 Computer Science Standards Grades 6-9 (ages 11-14)

#### 5.2 Level 2: Computer Science and Community (L2)

- ✦ Computational Thinking (CT)
  3. Define an algorithm as a sequence of instructions that can be processed by a computer.
  6. Describe and analyze a sequence of instructions being followed (e.g., describe a character's behavior in a video game as driven by rules and algorithms).
- ✦ Collaboration (CL)
  3. Collaborate with peers, experts, and others using collaborative practices such as pair programming, working in project teams, and participating in group active learning activities.